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amsterdam 1972

stichting
mathematisch
centrum

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2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

HOMOTOPY CLASSIFICATION OF TYPE (Zq, 1) ANR and APPLICATION to PERIODIC ACTIONS on (I-D) SPACES

Raymond Y.T. Wong *)

1. The purpose of this paper is to prove a homotopy classification theorem (Theorem 1) for ANR and to outline some of its consequences which, using a different lemma, are results already mentioned in [12]. Let Zq denote the integers modulo q, $q \geq 1$ ($Z_1 = \{0\}$). A connected. locally path connected metric space X is said to be an Eilenberg-Maclane space of type ($\mathbb{Z}q$, 1), or simply, of _type_ ($\mathbb{Z}q$, 1), provided the fundamental group $\pi_1(X)$ is isomorphic to $\mathbb{Z}q$ and $\pi_n(X) = \{0\}$ for all $n > 1$. Let E denote a fixed (but arbitrary) infinite-dimensional (I-D) normed linear space (NLS) which is homeomorphic ($\cong$) to F^ω or F^ω_f for some NLS F, where F^ω denotes the countable infinite product of F by itself and $F^\omega_f \subset F^\omega$ denotes the subset consisting of all points having at most finitely many non-zero coordinates. The following is our main theorem which classifies, up to homotopy type, all metric absolute neighborhood retracts (ANR) of type (Zq, 1).

Theorem 1. Let Y, Y' _be metrizable connected_ ANR _of type_ ($\mathbb{Z}q$, 1) _and let_ $e \in \pi_1(Y)$, $e' \in \pi_1(Y')$ _be generators. Then there is a homotopy_ equivalence $h : Y \to Y'$ such that $h_\#(e) = e'$.

The case for q = 1 is rather well known (see for example, the Collorary following Theorem 15 of Palais ([7])). This Theorem 1 may be viewed as a generalization of it. With this in mind, we assume from here on that $q > 1$. It is not well known that E-manifolds can be classified by their homotopy types ([4],[5]) and the same is true in the C^∞-category for separable C^∞- Hilbert manifolds ([3], [6]). We

*) Partially supported by NSF Grant GP20632. This research is prepared while the author is visiting the Mathematisch Centrum, Amsterdam.

Proposition 1.

(A) Each homotopy equivalence between E-manifolds is homotopic to a homeomorphism.

(B) Each homotopy equivalence between separable C^∞-Hilbert manifolds is homotopic to a C^∞-diffeomorphism.

Since all E-manifolds are ANR, applying Theorem 1 and Proposition 1 we obtain the following theorem which classifies all metrizable connected E-manifolds (or C^∞-Hilbert manifolds) of type $(\mathbb{Z}q, 1)$.

Theorem 2. (Classification) Let M and M_1 be metrizable E-manifolds of type $(\mathbb{Z}q, 1)$ and let $e \in \pi_1(M_1)$, $e_1 \in \pi_1(M_1)$ be generators. Then there is a homeomorphism $h : M \to M_1$ such that $h_\#(e) = e_1$.

Let l_2 denote the separable Hilbert space of all square summable complex sequences and S denote its unit sphere. For any $q > 1$, define a fixed point free periodic homeomorphism $\alpha : S \to S$ of period q by

$$\alpha(z_0, z_1, \ldots) = (e^{2\pi i/q} z_0, e^{2\pi i/q} z_1, \ldots).$$

Then α induces (by restrictions) periodic homeomorphisms $\alpha_n : S^{2n-1} \to S^{2n-1}$ of period q, where S^{2n-1} is the unit sphere of the 2n-dimensional complex space C^n. The inductive limit of $\{S^{2n-1}/\alpha_n\}_{n\geq 1}$ (S^{2n-1}/α_n the orbit spaces), denoted by $\varinjlim S^{2n-1}/\alpha_n$, is a CW-complex of type $(Zq, 1)$. Hence, by means of Theorem 1, we obtain

Theorem 3. Let M be a metrizable connected E-manifold of type $(\mathbb{Z}q, 1)$, then M has the same homotopy as $\varinjlim S^{2n-1}/\alpha_n$.

Let M be as above with $q > 1$ a prime number. The universal covering space $\tilde{M}$ of M is a homotopically trivial E-manifold such that the projection $p : \tilde{M} \to M$ is a q-folds covering map. By Proposition 1(A), $\tilde{M} \cong E$. Let $\beta : \tilde{M} \to \tilde{M}$ be any fixed point free period q homeomorphism (β always exists, see [9]). Then the orbit space $\tilde{M}/\beta$ is an E-manifold of type $(Zq, 1)$. By Theorem 2 there is a homeomorphism $h : \tilde{M}/\beta \to M$ which then induces a fibre homeomorphism $h_* : \tilde{M} \to \tilde{M}$. Let $\beta_* = h_* \circ \beta \circ h_*^{-1}$. We obtain the following theorem.

Theorem 4. (Representation) *Let M be a metrizable connected E-manifold of type* $(\mathbb{Z}q, 1)$, $q > 1$ *a prime number. Then there is a q-folds covering projection* $p : E \to M$ *and a fixed point free periodic homeomorphism* $\beta_*: E \to E$ *of period q such that* β_* *induces a homeomorphism* $\beta_0 : E/\beta_0 \to M$ *for which* $\beta_0 \circ p_0 = p \circ \beta_*$.

Added in proof. For the sake of completion we mention here that Theorem 1 is true for $q = 0$ ($\mathbb{Z}_0 = \mathbb{Z}$) and it is not difficult to show (using the universal covering space of Y and Lemma 1 of this paper) that Y has the homotopy type of a circle.

2. Aplication to periodic homeomorphisms and other results

Throughout this section let $q > 1$ denote a prime number.

Theorem 5. (Conjugation) *Let* $\beta, \beta_1 : E \to E$ *be fixed point free periodic homeomorphisms of period q. Then there is a homeomorphism* $h_0 : E \to E$ *such that* $h_0 \circ \beta = \beta_1 \circ h_0$.

Moreover, if $E = l_2$ *and* β, β_1 *are* C^∞*-smooth, we may choose* h_0 *to be a* C^∞*-diffeomorphism.*

Proof. The C^0 case. Let $b \in E$ and suppose $\lambda, \lambda_1 : ([0,1]) \to (E,b)$ are maps (preserving base points) such that $\lambda(1) = \beta(b)$ and $\lambda_1(1) = \beta_1(b)$. Let $p : E \to E/\beta$, $p_1 : E \to E/\beta_1$ denote the projections. Then $e = [p\circ\lambda] \in \pi_1(E/\beta)$ and $e_1 = [p_1\circ\lambda_1] \in \pi_1(E/\beta_1)$ are generators. It follows from theorem 2 that there is a homeomorphism $h : (E/\beta, p(b)) \to (E/\beta_1, p_1(b))$ such that $h_\#(e) = e_1$. The function h then induces a (fibre) homeomorphism $h_0 : (E,b) \to (E,b)$ such that $p_1 \circ h_0 = h \circ p$ and $h_0 \circ \beta(b) = \beta_1 \circ h_0(b)$. For each $x \in E$, since $\{h_0(x), h_0 \circ \beta(x)\} \subset p_1^{-1}(h\circ p(x))$, there is an $1 \le i \le q$ for which $h_0 \circ \beta(x) = \beta_1^i \circ h_0(x)$. Let $A_i = \{x \in E : h_0 \circ \beta(x) = \beta_1^i \circ h_0(x)\}$. We easily verify that each A_i is closed and $\{A_i\}$ are pairwise disjoint. Since E is connected and $A_1 \neq \emptyset$, hence $A_1 = E$. The C^∞ case follows exactly the same considerations using Theorem 1 and Proposition 1(B).

The above theorem is our principle application. In the following we state, without proof, several other consequences which are essentially corollaries of Theorem 5. We refer to [12] for their proofs. Suppose $X \cong X \times E$, a subset Y of X is said to be <u>E-deficient</u> if there is a homeomorphism $h : X \to X \times E$ such that $h(Y) \subset X \times \{0\}$. Let H denote the Hilbert space of all square complex sequences indexed by an <u>infinite</u> abstract set $I(H)$. Note that $H \cong H^{\omega}$ ([1]).

<u>Theorem 6.</u> (<u>Homeomorphism Extension</u>) <u>Let $A \subset H$ be a closed H-deficient subset. Then each period n homeomorphism $\beta : A \to A$ extends to a period n homeomorphism $\tilde{\beta}$ on H such that $\tilde{\beta}(x) = x$ if and only if $\beta(x) = x$.</u>

The proof of Theorem 6 is independent of Theorem 5 and is essentially an elementary application of [2.- Theorem 1]. Note that in [12 - Theorem 7] we assume n is a prime, which is irrelevant.

<u>Theorem 7.</u> (Closed Imbeddings) <u>Suppose X is a space which can be imbedded as a closed subset of a Hilbert space H. Then for any two fixed point free period q homeomorphisms β, β_1 on X, H respectively, there is a closed imbedding $m : X \to H$ satisfying $m \circ \beta = \beta_1 \circ m$.</u>

<u>Moreover, if X is a connected H-manifold, we may choose m so that $m(X)$ is a submanifold of H.</u>

<u>Theorem 8.</u> (<u>Negligible Subsets</u>) <u>Let $K_1, K_2, \ldots$ be closed H-deficient subsets of H. Suppose $\beta, \beta_1 : H \to H$ are fixed point free periodic homeomorphisms of period q for which $\beta(K) = K$, where $K = \bigcup_{i \geq 1} K_i$, then there is a homeomorphism: $H \to H \backslash K$ satisfying</u> $m \circ \beta = \beta_1 \circ m$.

For any space X, let $G(X)$ denote the space of homeomorphisms on X (of X onto itself) equipped with the compact-open topology. Note that $G(X)$ is a group under composition. Let $G_0(X)$ denote the subspace consisting of all periodic homeomorphisms and $G_n(X) = \{\beta \in G_0(X) : \text{period } (\beta) = n\}$.

Theorem 9. (Homeomorphism spaces are contractible) For $k \geq 0$, each $G_K(E)$ is contractible and there is a contraction $\{\phi_t\} : G(E) \to G(E)$ such that $\{\phi_t|_{G_k(E)}\}$, $k \geq 0$, is a contraction for $G_k(E)$.

In [12 - Corollary 3] it is proved that for $E \cong E^{\omega}$, the group $G(E)$ is simple, in the sense that $G(E)$ contains no non-trivial proper normal subgroup. For each fixed k, the collection of all finite composition of members in $G_k(E)$ clearly forms a non-trivial normal subgroup of $G(E)$. Hence we have

Theorem 10. (Periodic Stability) Suppose $E \cong E^{\omega}$. Then for any $h \in G(E)$ and any $k \geq 0$, there are $h_1, \dots, h_i \in G_k(E)$ such that $h = h_i \circ \dots \circ h_2 \circ h_1$.

3. Proof of Theorem 1

We say two maps $f,g : X \to Y$ are homotopic relative $A \subset X$, written $f \sim g$ rel (A), if there is a homotopy $\{\lambda_t\}$ joining f and g such that $\lambda_t(a) = \lambda_0(a)$ for all $a \in A$, $t \in [0,1]$. Let $\alpha : S \to S$ and $\alpha_n : S^{2n-1} \to S^{2n-1}$ be defined as before. To give a proof of Theorem 1, we need

Lemma 1. Let X, Y be metric spaces with X compact. Let $A \subset X$ be closed. Then for each map $g : X \to Y \times l_2$, there is a map $\tilde{g} : X \to Y \times l_2$ such that $\tilde{g} \sim g$ rel (A) and for $x \neq y$, $\tilde{g}(x) = \tilde{g}(y)$ only if $\{x,y\} \in A$.

Proof. (Technically we have to assume $g|_A$ is not one-to-one.) Note that the above statement implies $\tilde{g}|_A = g|_A$. Without loss of generality, we may write $Y \times l_2$ as $Y \times l_2 \times l_2 \times l_2$ and suppose $g(A) \subset Y \times l_2 \times \{0\} \times \{0\}$. Let $h : X \to l_2$ be an imbedding sucht that all coordinates of each $h(x)$ are positive. Let $\lambda : X \to [0,1]$ and $\lambda_1 : Y \times l_2 \to [0,1]$ be maps satisfying $\lambda^{-1}(0) = A$ and $\lambda_1^{-1}(0) = g(A)$. Define $\tilde{g} : X \to (Y \times l_2) \times l_2 \times l_2$ by $\tilde{g}(x) = (g(x), \lambda(x)h(x), \lambda_1(g(x))h(x))$. By the linear structure on l_2, $\tilde{g} \sim g$ rel(A).

<u>Lemma 2.</u> <u>(The Key Lemma)</u> <u>Let</u> X <u>be a metric</u> AR. <u>Suppose for some metric</u> ANR Y, <u>there is a</u> q-<u>fold covering projection</u> $p : X \to Y \times l_2$. <u>Let</u> $\lambda : ([0,1], 0) \to (Y \times l_2, b)$ <u>be a map such that</u> $[\lambda]$ <u>generates</u> $\pi_1(Y \times l_2, b)$. <u>Denote the lifting</u> $([0,1], 0) \to (X, b_0)$ <u>by</u> $\tilde{\lambda}$. <u>Let</u> $b_1 = \tilde{\lambda}(1)$. <u>Then there are imbeddings</u> $f_n : (S^{2n-1}, a_0) \to (X, b_0)$ <u>such that</u> (1) $f_1 \circ \alpha_1(a_0) = b_1$, (2) <u>for all</u> $n \geq 1$, $f_{n+1}|_{S^{2n-1}} = f_n$ <u>and</u> (3) $p \circ f_n(x) = p \circ f_n \circ \alpha_n(x)$ <u>for all</u> x.

<u>Proof.</u> Exactly the same as Lemma 1 of [12]. Note that the setting in [12] is for covering projection $p : E \to M$. We observe (1) the only property of E we need is E being an AR and (2) Lemma 2 of [12] may be replaced by Lemma 1 of this paper.

<u>Proof of Theorem 1.</u> Fix any $b \in Y$ and $a_0 \in S^1$. The universal covering space X of $Y \times l_2$ (with respect to base point $\tilde{b} = (b, 0)$) is a connected metrizable AR ([7 - Theorem 5 and 15]) for which the projection $p : X \to Y \times l_2$ is a q-folds covering map. Let $b_0 \in p^{-1}(\tilde{b})$ and let $\lambda : ([0,1], 0) \to (Y \times l_2, \tilde{b})$ be a map such that $[\lambda] = j_\#(e)$, where $j : Y \to Y \times \{0\} \subset Y \times l_2$ is the inclusion. λ lifts to a map $\tilde{\lambda} : ([0,1], 0) \to (X, b_0)$. Denote $b_1 = \tilde{\lambda}(1)$. Let $f_n : (S^{2n-1}, a_0) \to$ $\to (X, b_0)$ be imbeddings satisfying (1) - (3) of Lemma 2. $\{f_n\}$ induces (in a natural way) one-to-one maps $\tilde{f} : (\varinjlim S^{2n-1}, a_0) \to (X, b_0)$ and $f : \varinjlim (S^{2n-1}/\alpha_n) \to Y \times l_2$ satisfying $\tilde{f} \circ \alpha_1(a_0) = \tilde{b}_0$ and $p \circ \tilde{f} = f \circ p_0$, where $p_0 : \varinjlim S^{2n-1} \to \varinjlim S^{2n-1}/\alpha_n$ is the natural projection. It may be routinely verified that f is a weak homotopy equivalence. Hence by [7 - Theorem 14] and by Whitehead [10 - Theorem 1], f is a homotopy equivalence. Repeating the whole process for Y' we obtain the following diagram:

$$\begin{array}{ccccccccc}
 & & (X, b_0) & \xleftarrow{\tilde{f}} & (\varinjlim S^{2n-1}, a_0) & \xrightarrow{\tilde{f}'} & (X', b_0') & & \\
 & & \downarrow p & & \downarrow p_0 & & \downarrow p' & & \\
(Y, b) & \xrightarrow{j} & (Y \times l_2, b) & \underset{g}{\overset{f}{\rightleftarrows}} & (\varinjlim S^{2n-1}/\alpha_n, a) & \xrightarrow{f} & (Y' \times l_2, \tilde{b}') & \underset{j_1}{\overset{j'}{\leftrightarrows}} & (Y', b')
\end{array}$$

where g, j_1 are respectively homotopy inverses of f and j' (j_1 being obtained by shrinking l_2 to 0). In particular, $\tilde{f}'$ satisfies $\tilde{f}' \circ \alpha_1(a_0) = \tilde{b}_0'$, where $\tilde{b}_0'$ is the end point $\tilde{\lambda}'(1)$ of a map $\tilde{\lambda}' : ([0,1], 0) \to (X', b_0')$ for which $[p' \circ \tilde{\lambda}'] = j'_{\#}(e')$. Let $h = j_1 \circ f' \circ g \circ j$. Then
$h \circ \lambda = j_1 \circ f' \circ g \circ j \circ \lambda = j_1 \circ f' \circ g \circ p \circ \tilde{\lambda} \sim j_1 \circ f' \circ g \circ p \circ \lambda 1$,
where $\lambda_1 : ([0,1], 0) \to (X, b_0)$ is a map such that $\lambda_1 \sim \tilde{\lambda}$ rel (0,1) and $\lambda_1([0,1]) \subset f(\lim S^{2n-1})$. Thus
$h \circ \lambda \sim j_1 \circ f' \circ g \circ f \circ p_0 \circ \tilde{f}^{-1} \circ \lambda_1 \sim j_1 \circ f' \circ p_0 \circ \tilde{f}^{-1} \circ \lambda_1 =$
$= j_1 \circ p' \circ \tilde{f}' \circ \tilde{f}^{-1} \circ \lambda_1$. Since $\tilde{f}' \circ \tilde{f}^{-1}(\tilde{b}_0) = \tilde{b}_0'$, it follows that $h_{\#}(e) = e'$.

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